MATH 1A − QUIZ 9 − SOLUTIONS

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(1) *(5 points)* Sketch the graph of $f(x) = \frac{\sin(x)}{1+\cos(x)}$

$\boxed{\text{S}}$ Notice that f is periodic of period 2π, hence **from now on we will only focus on** $[0, 2\pi]$.

Moreover:

$$f(-x) = \frac{\sin(-x)}{1+\cos(-x)} = \frac{-\sin(x)}{1+\cos(x)} = -f(x)$$

Hence f is odd.

$\boxed{\text{D}}$ We want $1 + \cos(x) \neq 0 \Rightarrow \cos(x) \neq -1 \Rightarrow x \neq \pi + 2\pi m$ (where m is an integer), hence $\boxed{Dom = \mathbb{R} - \{\pi + 2\pi m\}}$ (in other words, everything except odd multiples of π)

$\boxed{\text{I}}$ $y-$intercept $f(0) = \frac{0}{2} = 0$, $x-$intercept: $f(x) = 0 \Leftrightarrow \frac{\sin(x)}{1+\cos(x)} = 0 \Leftrightarrow \sin(x) = 0 \Leftrightarrow x = \pi m$, where m is an integer.

HOWEVER, notice that for odd m, πm is **NOT** in the domain of f, hence the $x-$intercepts are $\boxed{x = 2\pi m}$, where m is an integer (that is, even multiples of π)

$\boxed{\text{A}}$ No Horizontal or Slant Asymptotes because f is periodic.

$$\lim_{x\to\pi^-} \frac{\sin(x)}{1+\cos(x)} \stackrel{H}{=} \lim_{x\to\pi^-} \frac{\cos(x)}{-\sin(x)} = \frac{-1}{-0^+} = \infty$$

$$\lim_{x\to\pi^+} \frac{\sin(x)}{1+\cos(x)} \stackrel{H}{=} \lim_{x\to\pi^+} \frac{\cos(x)}{-\sin(x)} = \frac{-1}{-0^-} = -\infty$$

Hence $\boxed{x = \pi}$ is a vertical asymptote (and more generally $x = \pi + 2\pi m$)

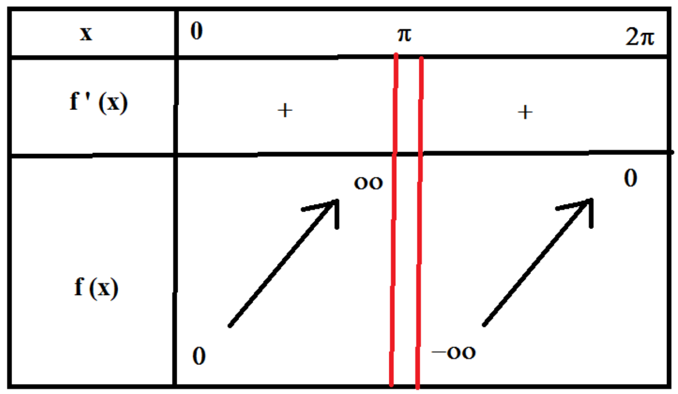
$\boxed{\text{I}}$

$$f'(x) = \frac{\cos(x)(1+\cos(x)) - \sin(x)(-\sin(x))}{(1+\cos(x))^2} = \frac{\cos(x) + \cos^2(x) + \sin^2(x)}{(1+\cos(x))^2} = \frac{1+\cos(x)}{(1+\cos(x))^2} = \frac{1}{1+\cos(x)}$$

Date: Friday, November 8th, 2013.

Drawing a sign table, we get:

1A/Math 1A - Fall 2013/Quizzes/Quiz9table1.png

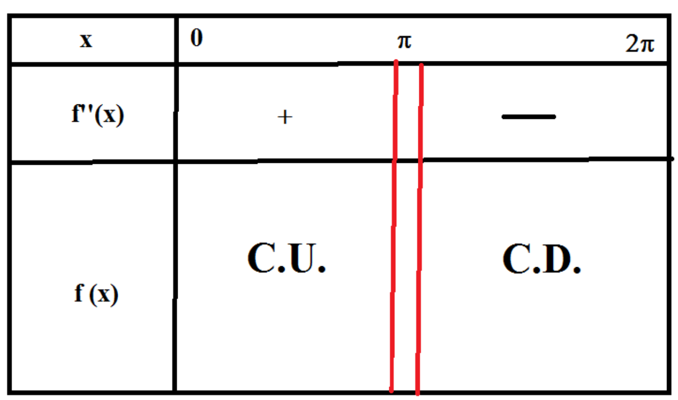


Therefore f is increasing on $(0, \pi)$ and on $(\pi, 2\pi)$. No local max/min

$\boxed{\text{C}}$ $f''(x) = \frac{-(-\sin(x))}{(1+\cos(x))^2} = \frac{\sin(x)}{(1+\cos(x))^2}$

Drawing a sign table, we get:

1A/Math 1A - Fall 2013/Quizzes/Quiz9table2.png

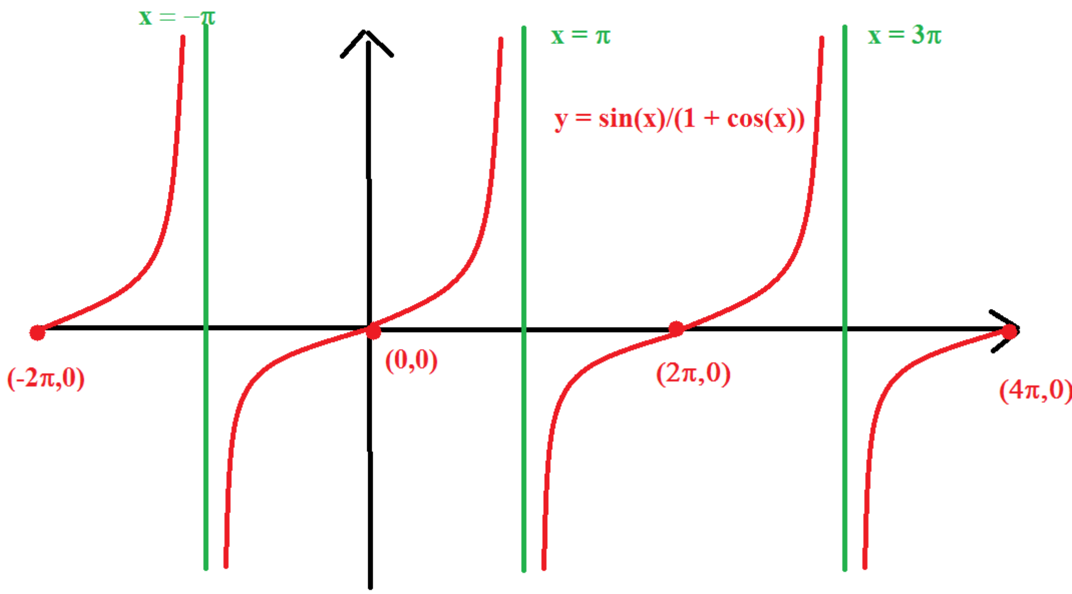


Therefore f is concave up on $(0, \pi)$ and concave down on $(\pi, 2\pi)$; Inflection point $\boxed{(0, 0)}$ (or more generally $(2\pi m, 0)$)

Note: This is not obvious from the table, but if you look at your graph, you'll notice this! The true reason is that f does change concavity at 0 and at 2π and at 4π etc (which your table does not reflect)

Hence, the graph of f looks like:

1A/Math 1A - Fall 2013/Quizzes/Quiz9graph.png



(2) *(1 point)* Solve the differential equation $x\,(y') + y = \sec^2(x)$.

Notice that the left-hand-side is precisely $(xy)'$ (by the product rule), and hence the differential equation becomes $(xy)' = \sec^2(x)$

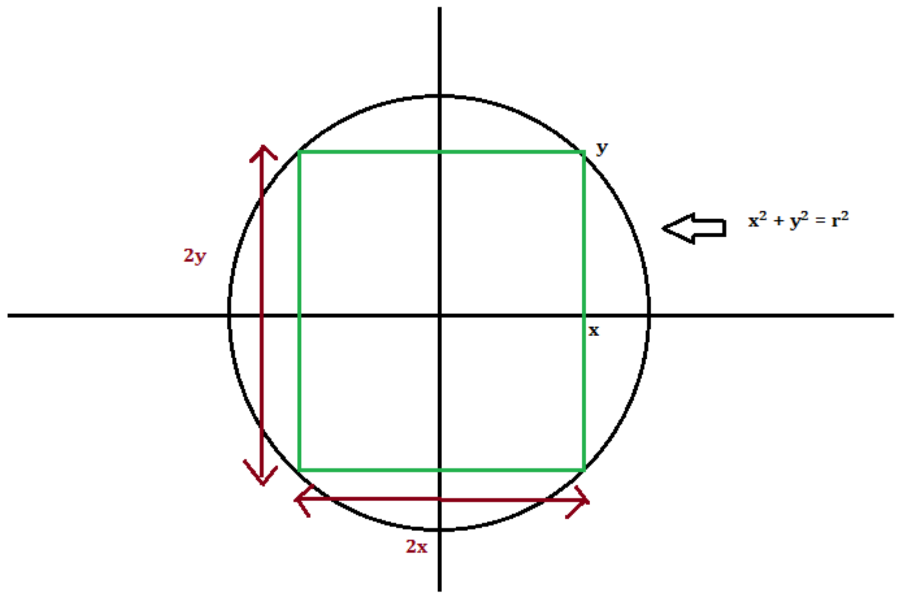
Antidifferentiating, we get: $xy = \tan(x) + C$

Hence: $\boxed{y = \frac{\tan(x)+C}{x}}$

(3) *(4 points)* What is the area of the largest rectangle that can be inscribed (put in) a circle of radius R?

(1) First draw a good picture! (here r is meant to be R)

1A/Math 1A - Fall 2013/Quizzes/Quiz9Rectangle.png



(2) Based on your picture, the length of the rectangle is $2x$ and the width is $2y$, and the area is $A = (2x)(2y) = 4xy$. But since (x, y) is on the circle, $x^2 + y^2 = R^2$, so $y = \sqrt{R^2 - x^2}$, so $A(x) = 4x\sqrt{R^2 - x^2}$. But instead of maxizing A, let's maximize:

$$f(x) = A^2 = 16x^2(R^2 - x^2) = 16x^2R^2 - 16x^4$$

(3) The constraint is $0 \leq x \leq R$

(4) $f'(x) = 32R^2x - 64x^3 = 32x(R^2 - 2x^2) = 0$

Which gives $x = 0$ or $x^2 = \frac{R^2}{2}$, hence: $x = \frac{R}{\sqrt{2}}$ (we're ignoring $x = 0$ and $x = -\frac{R}{\sqrt{2}}$ here)

Also, $f(0) = f(R) = 0$, and:

$$f\left(\frac{R}{\sqrt{2}}\right) = 16\left(\frac{R}{\sqrt{2}}\right)^2\left(R^2 - \left(\frac{R}{\sqrt{2}}\right)^2\right) = 16\frac{R^2}{2}\left(R^2 - \frac{R^2}{2}\right) = 8R^2 \times \frac{R^2}{2} = 4R^4 > 0$$

So by the closed interval method, $f\left(\frac{R}{\sqrt{2}}\right) = 4R^4$ is the absolute maximum of f.

And hence the largest area is $\boxed{A = \sqrt{4R^4} = 2R^2}$ (remember that we squared A to get f, so we have to take square root of f to get A back).

Note: Incidentally, the optimal rectangle is a square! (but you did not need to know that to solve the problem)